Influences of Lorentz symmetry violation on charged Dirac fermions in cosmic dislocation space-time

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Abstract

In this paper, the relativistic and non-relativistic behavior of charged Dirac fermions is investigated in the presence of a Kratzer-like potential under the influence of a broken Lorentz symmetry in space-time with the cosmic screw dislocation background. The effect of the Lorentz symmetry breaking is defined by a fixed vector field. We indicate the interaction of a charged Dirac fermion with a magnetic field determined by an electromagnetic field tensor in the background involving a Kratzer-like potential generated by the Lorentz symmetry violation effect. In this way, the energy eigenvalues and relevant wave functions of the generalized Dirac and Schrödinger-Pauli equation in the presence of the induced Kratzer-like potential are obtained by an analytical method.

Keywords: Lorentz symmetry violation; Cosmic screw dislocation; Charged Dirac fermion; Generalized Dirac equation; Generalized Schrödinger-Pauli equation; Kratzer-like potential.
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1 Introduction

During the past three decades Lorentz-invariance breaking mechanisms have extensively been discussed in the literature. This was inspired by the initial work of Kostelecký and Samuel [1, 2] and the work by Carroll, Field and Jackiw [3]. Whereas in Ref. [1, 2] it was shown that Lorentz invariance could spontaneously be broken in string theories, the approach in Ref. [3] discusses the $1+3$-dimensional Maxwell Lagrangian with an additional Chern-Simons term, which explicitly breaks the Lorentz invariance but gauge invariance remains as a symmetry. These papers have triggered further investigations of Lorentz and CPT violations in electrodynamics [4, 5] and in the standard model of particle physics [6–8]. See also the series of meetings [9]. For a summary of theoretical backgrounds and experimental tests of Lorentz invariance see Ref. [10]. A recent discussion related to quantum gravity can be found in Ref. [11].

Being a bit more specific, the Lorentz symmetry violation (LSV) effects and their possible experimental observations have, for example, been looked at in [12], where the neutron spin coupling to a Lorentz- and CPT-violating background field is considered using a magnetometer with overlapping ensembles of K and ^{3}He atoms. A systematic theoretical investigation to identifying atoms which show the greatest promise for detecting a Lorentz symmetry violation in the electron-photon sector is discussed in [13]. A search for Local Lorentz invariance violation associated with operators of mass dimension $d = 6$ in the pure-gravity sector with short-range gravitational experiments is, for example, presented in [14]. Let us also mention that a formalism for analyzing short-range tests of gravity for general signals of Lorentz violation is presented in [15], and a study on nonminimal Standard-Model-Extension effects arising from particle and antiparticle charge-to-mass ratio measurements in Penning traps can be found in Ref. [16].

Particular interest has been and still is in the coupling of charged as well as neutral fermions in a non-minimal way via the scheme $D_\mu = \partial_\mu - qA_\mu - g\mathrm{b}^\nu G_{\mu\nu}$. Here, b^μ is a fixed four-vector that affects a vector field which causes breaking of the Lorentz symmetry, and $G_{\mu\nu}$ stands either for the electromagnetic field tensor $F_{\mu\nu}$ or its dual $F^\star_{\mu\nu} = \varepsilon_{\mu\nu\rho\sigma}F^{\rho\sigma}$. It has been shown that such LSV interactions may result in an observable difference of the Aharonov–Casher effect between particles and their anti-particle [17–19]. Implications of such interactions on the scalar Aharonov-Bohm effect and other geometric phases have also been studied extensively [20, 21]. See also Ref. [22], where the Aharonov-Bohm-Casher problem is studied in the additional presence of a cosmic string background.

In this work, according to an approach beyond the standard model of particle physics, we study the influence of the LSV on the relativistic and non-relativistic behavior of a charged Dirac fermion, which interacts with a uniform magnetic field in cosmic string space-time with the screw dislocation [23,24]. In this way, we solve the generalized Dirac equation in the presence of a Kratzer-like potential in the relativistic and non-relativistic limit. In both cases, the Kratzer-like potential stems from a common LSV scenario in the presence of the cosmic screw dislocation in the background space-time. Meanwhile, this LSV scenario is created by a fixed space-like vector field in the radial direction.

This paper is organized as follows. In the next section, we set up the stage starting with a line element which corresponds to a background space-time provided by a cosmic screw dislocation. Then, in order to investigate the interaction of a relativistic charged Dirac fermion with a uniform magnetic field under the background involving a Kratzer-like potential induced by the possible scenario of the LSV effect, we incorporate a relevant coupling into the generalized Dirac equation, which includes an electromagnetic four-vector term potential and a term associated with breaking the Lorentz symmetry. In this way, after assigning a uniform magnetic field in the direction of the screw dislocation and a fixed space-like vector field in radial direction perpendicular to the magnetic field direction, we find a solvable form of the relativistic limit of the generalized Dirac equation. We then utilise the Nikiforov–Uvarov (NU) method [25] to obtain exact analytical solutions. In Sect. 3, we study the non-relativistic behavior of the charged Dirac fermions in the background having a Kratzer-like potential generated by the possible scenario of the LSV effect in the presence of the cosmic screw dislocation in space-time. We obtain the Schrödinger-Pauli equation and find exact analytical solutions similar to the relativistic case. We conclude in Sect. 4 with a short summary and some additional remarks.

2 Relativistic limit of the generalized Dirac equation under the LSV effect

In this section, we first construct the Dirac equation in the relativistic regime with a background space-time defined by a cosmic screw dislocation and then provide exact analytical solutions. In this way, we observe that the corresponding charged Dirac fermion is affected by a Kratzer-like potential induced by a possible scenario of the LSV effect. Accordingly, we need to embed a relevant coupling containing an electromagnetic four-vector term potential and a term associated with breaking the Lorentz symmetry in the generalized Dirac equation.

To begin with, let us describe the space-time containing curvature and torsion via the following line element corresponding to the cosmic screw dislocation background in the $1+3$-dimensional cylindrical coordinate [26–38]

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + \mathrm{d}r^2 + \alpha^2 r^2 \mathrm{d}\varphi^2 + (\mathrm{d}z + \chi \mathrm{d}\varphi)^2 \,. \tag{2.1}$$

Here t stands for the time coordinate, $t \in \mathbb{R}^+$ and (r, φ, z) are the usual cylinder coordinates in $\mathbb{R}^3$ taking values in the ranges $r \in \mathbb{R}^+$, $\varphi \in [0, 2\pi]$ and $z \in \mathbb{R}$. Moreover, we will work in units where $\hbar = 1$ and $c = 1$.

In the above we have also introduced two parameters, α and χ, characterizing the curvature and torsion of the space-time. The first parameter α, which is smaller than unity, is expressed as $\alpha = 1 - 4\varpi G$, with ϖ being the linear mass density related to the cosmic string and G is Newton's gravitational constant [39].

The second parameter χ in essence represents the modulus of Burgers vector $\vec{b} = b\vec{e}_z$, which is taken in the z-direction so that $\chi = b/2\pi$. In the context of linear topological defects, the Burgers vector characterizes a line defect. More precisely, it is a screw dislocation as by a full rotation about the z-axis it results in a translation in the z-direction by $\vec{b}$. In principle b is an arbitrary real number. For a graphical presentation of the space characterised by α and χ we refer to the figures in Ref. [37].

With the line element (2.1), we can associate a contravariant metric tensors $\mathrm{g}^{\mu\nu}$ which explicitly reads

$$\mathrm{g}^{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{1}{\alpha^2 r^2} & -\frac{\chi}{\alpha^2 r^2} \\ 0 & 0 & -\frac{\chi}{\alpha^2 r^2} & 1 + \frac{\chi^2}{\alpha^2 r^2} \end{pmatrix} . \tag{2.2}$$

Having setup the space-time structure, we will now derive within this framework the Dirac equation for charged spin one-half fermions minimally coupled to an electromagnetic four-vector potential and non-minimally to the electromagnetic four-tensor breaking the Lorentz symmetry by the following scheme

$$\mathrm{i}\gamma^\mu \nabla_\mu \to \mathrm{i}\gamma^\mu \nabla_\mu - q\gamma^\mu A_\mu - g\mathrm{b}^\mu F_{\mu\nu}\gamma^\nu . \tag{2.3}$$

Hence, the generalized Dirac equation for a charged fermion with mass m in this configuration reads

$$[\mathrm{i}\gamma^\mu \nabla_\mu - q\gamma^\mu A_\mu - g\mathrm{b}^\mu F_{\mu\nu}\gamma^\nu - m]\, \Psi\,(t, \vec{r}) = 0. \tag{2.4}$$

Here, the parameter q denotes the electric charge of the fermions and A_μ represents the electromagnetic four-vector potential such that $A_\mu = (A_0, \vec{A})$. The third term of the left-hand side of Eq. (2.4) represents the LSV effect in the geometric approach, where b^μ is considered as a fixed four-vector coupled to the electromagnetic field tensor $F_{\mu\nu}(x)$ via the coupling constant g [40–42]. We recall that the electric and magnetic field components are given in the usual way as $F_{0i} = -F_{i0} = -E_i$ and $F_{ij} = \epsilon_{ijk}B^k$, with $i, j = 1, 2, 3$. It is worth mentioning that multiplying the spatial part of the fixed four-vector b^μ, which is responsible for the Lorentz symmetry breaking, by the coupling constant g induces a type of electric dipole moment $\vec{\mathrm{d}} = g\vec{\mathrm{b}}$ (analogous to a permanent electric dipole moment) such that it is fixed by the background [18, 20, 22]. Besides, the generalized Dirac matrices are given by $\gamma^\mu = e^\mu{}_a(x)\gamma^a$, where the objects $e^\mu{}_a(x)$ are the so-called inverse of the tetrads, and γ^a denotes the standard Dirac matrices corresponding to a flat space-time by given [43][1]

$$\gamma^0 = \hat{\beta} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \gamma^i = \hat{\beta}\hat{\alpha}^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}, \quad \Sigma^i = \begin{pmatrix} \sigma^i & 0 \\ 0 & \sigma^i \end{pmatrix}. \tag{2.5}$$

The spin vector and the Pauli matrices are denoted by $\vec{\Sigma}$ and σ^i, respectively. Moreover, the Pauli matrices obey the well-known anti-commutation relations $\{\sigma^i, \sigma^j\} = 2\delta_{ij}\mathrm{I}$, where δ_{ij} denotes the Kronecker delta and I denotes the 2×2 identity matrix, which may also be written as $\sigma^0 = \mathrm{I}$.

Having this in mind the components of the generalized Dirac matrices are found to be given by

$$\gamma^t = \gamma^0, \qquad \gamma^r = \gamma^1, \qquad \gamma^\varphi = (1/\alpha r)\gamma^2, \qquad \gamma^z = -(\chi/\alpha r)\gamma^2 + \gamma^3. \tag{2.6}$$

In a next step we need to construct the covariant derivatives ∇_μ being expressed in the form $\nabla_\mu = \partial_\mu + \Gamma_\mu$, where Γ_μ represents the spinorial connections. The components of these connections are found via the relation $\Gamma_\mu = \frac{\mathrm{i}}{4}\omega_{\mu ab}\Sigma^{ab}$, where Σ^{ab} is given by $\Sigma^{ab} = \frac{\mathrm{i}}{2}\left[\gamma^a, \gamma^b\right]$.

However, before discussing the spin connection $\omega_{\mu ab}$ mentioned in above relation, we need to address the necessity of using the spinor theory in defected space-time background. It is not surprising that spinors need to be represented locally by using the transformation laws under the background of the defected space-time. These transformation laws are raised from local Lorentz transformations [44]. Now we set up a local reference frame by a non-coordinate basis in the form $\hat{\theta}^a = e^a{}_\mu(x)\mathrm{d}x^\mu$. Here the tetrads $e^a{}_\mu(x)$ are in essence defined via their inverse obeying the relation $\mathrm{d}x^\mu = e^\mu{}_a(x)\hat{\theta}^a$. Note that $\mathrm{d}x^\mu$ are the so-called coordinate basis of 1-forms.

Thus, according to this formalism and the line element (2.1), we can set the non-coordinate basis related to the local reference frame as $\hat{\theta}^0 = \mathrm{d}t$; $\hat{\theta}^1 = \mathrm{d}r$; $\hat{\theta}^2 = \alpha r\,\mathrm{d}\varphi$; $\hat{\theta}^3 = \mathrm{d}z + \chi\,\mathrm{d}\varphi$. Furthermore, the tetrads and their inverse must satisfy the conditions $e^a{}_\mu(x)\,e^\mu{}_b(x) = \delta^a{}_b$ and $e^\mu{}_a(x)\,e^a{}_\nu(x) = \delta^\mu{}_\nu$.[2] Moreover, the tetrads and metric tensors satisfy the relation $\mathrm{g}_{\mu\nu}(x) = e^a{}_\nu(x)e^b{}_\nu(x)\eta_{ab}$, where the Minkowski tensor η_{ab} is given by $\eta_{ab} = \mathrm{diag}(- + ++)$.

As a result the non-vanishing components of the inverse tetrad $e^\mu{}_a(x)$ are given by $e^t{}_0(x) = e^r{}_1(x) = e^z{}_3(x) = 1$, $e^\varphi{}_2(x) = 1/\alpha r$ and $e^z{}_2(x) = -\chi/\alpha r$. In order to calculate the non-vanishing components of the spinorial connection Γ_μ, we first need to find the components of the spin connection. These can be obtained by solving the Maurer-Cartan structure equations in the form $\mathrm{d}\hat{\theta}^a + \omega^a{}_b \wedge \hat{\theta}^b = 0$ written in the absence of the torsion, with $\omega^a{}_b = \omega_\mu{}^a{}_b(x)\mathrm{d}x^\mu$. Thereby, the non-null components of the spin connection are found as $\omega_\varphi{}^2{}_1(x) = -\omega_\varphi{}^1{}_2(x) = \alpha$. Thus, the only non-null component of the spinorial connection is

$$\Gamma_\varphi = -\frac{\mathrm{i}\alpha}{2}\Sigma^3. \tag{2.7}$$

In this way, in order to find the analytical solutions of Eq. (2.4), we first need to find closed-form expressions for the generalized Dirac matrices γ^μ and the associated spinorial connections Γ_μ. Thus, we get $\mathrm{i}\gamma^\mu\Gamma_\mu = \mathrm{i}\gamma^1/2r$. Then, we need to expand the third term of the left-hand side of equation (2.4), which corresponds to the LSV background, as follows

$$-g\mathrm{b}^\mu F_{\mu\nu}(x)\gamma^\nu \equiv -\gamma^t g\,\vec{\mathrm{b}} \cdot \vec{E} + \vec{\gamma} \cdot \left(g\,\mathrm{b}^0\vec{E} + g\,\left(\vec{\mathrm{b}} \times \vec{B}\right)\right), \tag{2.8}$$

[1] We use Latin indices for representing the local reference frame, i.e., $a, b = 0, 1, 2, 3$ and the spatial components of the local reference frame, i.e., $i, j = 1, 2, 3$.

[2] Here Greek indices are considered as $\mu, \nu = t, \rho, \varphi, z$.

in which $\vec{\gamma}$ is given by $\vec{\gamma}=(\gamma^r,\gamma^\varphi,\gamma^z)$, also b^0 and $\vec{\mathrm{b}}=(\mathrm{b}^1,\mathrm{b}^2,\mathrm{b}^3)$ are defined as time-like and fixed space-like vectors, respectively. Besides, the vector fields $\vec{E}$ and $\vec{B}$ are introduced as electric and magnetic fields in the local reference frame of observers. At this point, by focusing on the second and third terms of the left-hand side of Eq. (2.4), we can define the effective vector potential as $\vec{A}^{\mathrm{eff}}=\vec{A}-\frac{g}{q}[\mathrm{b}^0\vec{E}+(\vec{\mathrm{b}}\times\vec{B})]$ (note that in Eq. (2.8) we can see the expanded form of the term corresponding to the LSV effect). Hence, an effective magnetic field in this context can be written as $\vec{B}^{\mathrm{eff}}=\vec{\nabla}\times\vec{A}^{\mathrm{eff}}$. Meanwhile, the first term of the right-hand side of Eq. (2.8) gives rise to a geometric phase corresponding to an effect analogous to the scalar Aharonov–Bohm effect under the background of the LSV effect investigated by Bakke et al [20]. Then, to investigate a new scenario involving the interaction of an electric dipole moment induced by the LSV effect with a magnetic field, we propose a fixed space-like vector in the direction of the r coordinate and a uniform magnetic field in the direction of the screw dislocation in the following form [45,46]

$$\vec{\mathrm{b}}=\mathrm{b}^1\hat{r},\qquad \vec{B}=B_0\hat{z},\tag{2.9}$$

where the parameters b^1 and B_0 are constant. Thus, based on the uniform magnetic field given by Eq. (2.9), we can consider an electromagnetic three-vector potential in the background as

$$\vec{A}=\frac{B_0}{2}r\,\hat{\varphi}.\tag{2.10}$$

With all these ingredients, we can write down the generalized Dirac equation in an explicit form in the presence of the LSV effect in the corresponding background space-time as follows

$$\begin{aligned}&\left[\mathrm{i}\partial_t+\mathrm{i}\hat{\alpha}^1\left(\partial_r+\frac{1}{2r}\right)+\mathrm{i}\frac{\hat{\alpha}^2}{\alpha r}\left(\partial_\varphi-\chi\partial_z+\mathrm{i}\frac{qB_0}{2}r+\mathrm{i}g\mathrm{b}^1B_0\right)\right.\\&\left.+\mathrm{i}\hat{\alpha}^3\partial_z-\hat{\beta}m\right]\Psi\left(t,\vec{r}\right)=0.\end{aligned}\tag{2.11}$$

It is evident that Eq. (2.11) is established by considering Eq. (2.8) in terms of the fields given by Eq. (2.9). In addition, the effects associated with the generalized Dirac matrices (2.6), the non-null component of the spinorial connection (2.7) and the vector potential (2.10) can be well seen in the second and third terms of Eq. (2.11). Furthermore, the impact of the fixed four-vector becomes obvious through the appearance of the coupling constant g in the third term of Eq. (2.11).

In the following, to represent the generalized Dirac equation (2.11) in the form of two coupled equations, we express the Fermi field $\Psi\left(t,\vec{r}\right)$ in terms of two 2-spinors $\psi_1\left(t,\vec{r}\right)$ and $\psi_2\left(t,\vec{r}\right)$ as follows, $\Psi\left(t,\vec{r}\right)=\left(\psi_1\left(t,\vec{r}\right),\psi_2\left(t,\vec{r}\right)\right)^{\mathrm{T}}$. Here the spinors ψ_1 and ψ_2 are considered as two-component spinors which may be taken as pair-wise eigenstates of the third Pauli matrix, that is, $\sigma^3\psi_1^{\pm}=\pm\psi_1^{\pm}$ and $\sigma^3\psi_2^{\pm}=\pm\psi_2^{\pm}$. Therefore, by replacing the new form of the Fermi field in the generalized Dirac equation (2.11), we arrive at

$$\begin{aligned}\left(\mathrm{i}\partial_t-m\right)\psi_1\left(t,\vec{r}\right)=&\left[-\mathrm{i}\sigma^1\left(\partial_r+\frac{1}{2r}\right)-\mathrm{i}\frac{\sigma^2}{\alpha r}\left(\partial_\varphi-\chi\partial_z+\mathrm{i}\frac{qB_0}{2}r+\mathrm{i}g\mathrm{b}^1B_0\right)\right.\\&\left.-\mathrm{i}\sigma^3\partial_z\right]\psi_2\left(t,\vec{r}\right),\end{aligned}\tag{2.12a}$$

$$\begin{aligned}\left(\mathrm{i}\partial_t+m\right)\psi_2\left(t,\vec{r}\right)=&\left[-\mathrm{i}\sigma^1\left(\partial_r+\frac{1}{2r}\right)-\mathrm{i}\frac{\sigma^2}{\alpha r}\left(\partial_\varphi-\chi\partial_z+\mathrm{i}\frac{qB_0}{2}r+\mathrm{i}g\mathrm{b}^1B_0\right)\right.\\&\left.-\mathrm{i}\sigma^3\partial_z\right]\psi_1\left(t,\vec{r}\right).\end{aligned}\tag{2.12b}$$

In order to solve these two coupled equations, we combine them by removing the spinor ψ_2 in the first equation via the second equation in (2.12). This leads us to [3]

$$
\begin{aligned}
\left(\partial_t^2+m^2\right)\psi_1 = & \left[\partial_r^2+\frac{1}{r}\partial_r+\frac{1}{\alpha^2 r^2}\left(\partial_\varphi-\chi\partial_z\right)^2+\partial_z^2\right]\psi_1 \\
& -\frac{1}{4r^2}\psi_1+\frac{2\mathrm{i}}{\alpha^2 r^2}\left(\frac{qB_0}{2}r+g\mathrm{b}^1 B_0\right)\left(\partial_\varphi-\chi\partial_z\right)\psi_1 \\
& -\frac{1}{\alpha^2 r^2}\left(\frac{qB_0}{2}r+g\mathrm{b}^1 B_0\right)^2\psi_1-\frac{\mathrm{i}\sigma^3}{\alpha r^2}\left(\partial_\varphi-\chi\partial_z\right)\psi_1 \\
& +\frac{\sigma^3}{\alpha r^2}\left(\frac{qB_0}{2}r+g\mathrm{b}^1 B_0\right)\psi_1-\frac{qB_0\sigma^3}{2\alpha r}\psi_1.
\end{aligned} \tag{2.13}
$$

Obviously spinor ψ_1 may be chosen as an eigenstate of σ^3 corresponding to the two eigenvalues ± 1. Hence, we set $\sigma^3\psi_1^s = s\psi_1$ with $s=\pm 1$. Furthermore, the Hamilton operator $\hat{H}$, in essence given by right-hand side of Eq. (2.13), is invariant under translations along the z-axis as well as under rotations about the same axis. That is, it commutes with the linear momentum operator $\hat{p}_z=-\mathrm{i}\partial_z$ and the angular momentum operator $\hat{J}_z=\hat{L}_z+\hat{S}_z=-\mathrm{i}\partial_\varphi+\frac{s}{2}$, $[\hat{H},\hat{p}_z]=0=[\hat{H},\hat{J}_z]$. As a result we may choose the following ansatz

$$
\psi_1^s\left(t,\vec{r}\right)=\mathrm{e}^{-\mathrm{i}\mathcal{E}t+\mathrm{i}(\ell+\frac{1}{2})\varphi+\mathrm{i}kz}\Phi_s(r), \tag{2.14}
$$

where $\mathcal{E}$ denotes the energy eigenvalue of the underlying Dirac Hamiltonian as the system is invariant under translations in time. Furthermore, the wave number $k\in\mathbb{R}$ and angular momentum quantum number $\ell\in\mathbb{Z}$ denote the eigenvalues of operators $\hat{p}_z$ and $\hat{L}_z$, respectively. Now, if we substitute $\psi_1^s\left(t,\vec{r}\right)$ given in Eq. (2.14) into Eq. (2.13), we arrive at the second-order differential equation in terms of a new wave function $\Phi_s(r)$ as follows

$$
\begin{aligned}
&\frac{\mathrm{d}^2\Phi_s(r)}{\mathrm{d}r^2}+\frac{1}{r}\frac{\mathrm{d}\Phi_s(r)}{\mathrm{d}r}+\frac{1}{r^2}\Bigg[-\left(k^2+m^2+\frac{q^2B_0^2}{4\alpha^2}-\mathcal{E}^2\right)r^2 \\
&-\frac{qB_0\,\xi_\ell}{\alpha^2}r-\left(\frac{\xi_\ell^2}{\alpha^2}-\frac{s\,\xi_\ell}{\alpha}+\frac{1}{4}\right)\Bigg]\Phi_s(r)=0,
\end{aligned} \tag{2.15}
$$

where ξ_ℓ is defined as

$$
\xi_\ell=\ell+\frac{1}{2}-\chi k+g\mathrm{b}^1 B_0. \tag{2.16}
$$

In the literature, Eq. (2.15) is recognized as the NU equation [25]. Now, when focusing on Eq. (2.15), we see that the following expression, representing the LSV scenario generated by the fixed space-like vector in Eq. (2.9), takes the form of a Kratzer potential.

$$
\mathcal{V}(r)=-\frac{g\mathrm{b}^1 qB_0^2}{\alpha^2 r}-\left[\frac{\left(g\mathrm{b}^1B_0\right)^2}{\alpha^2}-\frac{g\mathrm{b}^1B_0}{\alpha}\left(s-\frac{2\left(\ell+\frac{1}{2}-\chi k\right)}{\alpha}\right)\right]\frac{1}{r^2}. \tag{2.17}
$$

In this Kratzer-like potential the influence of the LSV scenario and screw dislocation becomes transparent. It should be noted that the Kratzer potential (as a static scalar potential) has many applications in various fields of physics, such as quantum field theory, particle physics, molecular and solid-state physics [47–49]. For instance, in the context of molecular physics, the Kratzer potential has been considered to describe interactions in a molecular system; for instance, the interactions of a non-relativistic quantum particle with the Kratzer molecular potential in space-time background [50]. Besides, this potential has been considered in the investigation of anharmonic oscillatory systems [51].

[3] Because of the compactness of the following equation, we temporarily drop the t and $\vec{r}$ dependencies in the spinor component $\psi_1\left(t,\vec{r}\right)$.

The Kratzer potential is amongst the most attractive physical potentials as it contains a degeneracy-removing inverse square term besides the common Coulomb term.

The Kratzer potential is a central potential involving a Coulomb-like term and an inverse-square. In order to be an attractive potential, it is sufficient that the Coulomb-like part is an attractive potential. According to Eq. (2.17), the term related to the Coulomb potential can remain an attractive potential by considering the positive values of $qB_0\,\xi_\ell$. To maintain the positivity of $qB_0\,\xi_\ell$, assuming our Dirac fermion being an electron with $q = -|q|$ and b^1 being always a positive constant [45], we need to adopt a negative value for $B_0\,\xi_\ell$. Another constituting term of the Kratzer-like potential (induced by the possible scenario of the LSV effect in the presence of the cosmic screw dislocation in the background space-time) is the inverse square potential used in many problems such as the investigation of the Efimov effect [52, 53], conformal invariance [54], as a singular potentials [55] in the presence of the hyperspherical approximation [56] as well as inducing the inverse square potential under the Aharonov–Casher effect [57]. To analyze the quantum dynamics of charged Dirac fermions in the background space-time described by the cosmic screw dislocation under the influence of the LSV effect in the relativistic regime, we need to find exact analytical solutions of the second-order differential equation (2.15). For this purpose, according to the form of Eq. (2.15), we can apply the NU method [25], which results in explicit expressions for wave functions and corresponding energy eigenvalues of the Schrödinger and Schrödinger-like equations. Based on this method, the solutions for Eq. (2.15), which in essence can be reduced to a confluent hypergeometric equation, may be written as follows (see Ref. [58] for more details of the NU method)

$$\begin{aligned}\Phi^{\mathrm{R}}_{sn\ell}(r) =& \mathrm{N}^{R}_{n\ell}\, r^{\frac{1}{2}\sqrt{1+\frac{4\xi_\ell^2}{\alpha^2}-\frac{4s\xi_\ell}{\alpha}}}\, \mathrm{e}^{-\sqrt{k^2+m^2+\frac{q^2B_0^2}{4\alpha^2}-\left(\mathcal{E}^{\mathrm{R}}_{n\ell}\right)^2}\, r} \\ &\times \mathcal{L}_n^{\sqrt{1+\frac{4\xi_\ell^2}{\alpha^2}-\frac{4s\xi_\ell}{\alpha}}}\left(2\sqrt{k^2+m^2+\frac{q^2B_0^2}{4\alpha^2}-\left(\mathcal{E}^{\mathrm{R}}_{n\ell}\right)^2}\; r\right).\end{aligned} \tag{2.18}$$

Here $\mathrm{N}^{R}_{n\ell}$ and $\mathcal{L}_n^{(\alpha)}(x)$ denote the normalization constant and the generalized Laguerre polynomial, respectively. We added a superscript R in order to indicate that this result stands for the relativistic limit discussed in this section.

In addition, following the NU method, we also obtain the relativistic energy eigenvalues corresponding to Eq. (2.15)

$$\left(\mathcal{E}^{\mathrm{R}}_{sn\ell}\right)^2 = k^2+m^2+\frac{q^2B_0^2}{4\alpha^2} - \frac{q^2B_0^2\xi_\ell^2}{\alpha^4\left(1+2n+\sqrt{1+\frac{4\xi_\ell^2}{\alpha^2}-\frac{4s\xi_\ell}{\alpha}}\right)^2}. \tag{2.19}$$

The relativistic energy eigenvalues corresponding to a charged Dirac fermion in space-time with the cosmic screw dislocation under the background involving a Kratzer-like potential induced by the LSV scenario generated by the fixed space-like vector in Eq. (2.9) then read $\mathcal{E}^{\mathrm{R}}_{sn\ell} = \pm\sqrt{\left(\mathcal{E}^{\mathrm{R}}_{sn\ell}\right)^2}$. In this regard, one can see the interaction between the uniform magnetic field and a relativistic charged Dirac fermion with an induced analogous electric dipole moment. Note that this interaction is implicitly given by the parameter ξ_ℓ as defined in Eq. (2.16). Furthermore, it is observed that Eq. (2.19) depends on other parameters such as the quantum numbers n and ℓ, the parameter α associated with the deficit angle, the rest mass m related to the fermionic field, the wave number k and the eigenvalues s of σ^3.

In the next section, based on our geometric approach, we want to discuss the non-relativistic quantum dynamics of charged Dirac fermions under the influence of the Kratzer-like potential given by the LSV scenario determined by the fixed space-like vector in Eq. (2.9).

3 Non-relativistic limit of the generalized Dirac equation under the LSV effect

To investigate the non-relativistic behavior of charged Dirac fermions in the background space-time described by the cosmic screw dislocation in the presence of the Kratzer-like potential under the LSV scenario, we must first obtain the non-relativistic limit of the generalized Dirac equation. Following the standard procedure, we take out a phase factor $\exp\{-imt\}$ and introduce the large and small components of the Dirac spinor as follows

$$\Psi\left(t,\vec{r}\right)=\mathrm{e}^{-imt}\begin{pmatrix}\psi_1\left(t,\vec{r}\right)\\ \psi_2\left(t,\vec{r}\right)\end{pmatrix}. \tag{3.1}$$

Inserting this into (2.11) we arrive at the pair of coupled equation for the two 2-spinors as

$$\begin{aligned}\mathrm{i}\partial_t\psi_1\left(t,\vec{r}\right)=&\left[-\mathrm{i}\sigma^1\left(\partial_r+\frac{1}{2r}\right)-\mathrm{i}\frac{\sigma^2}{\alpha r}\left(\partial_\varphi-\chi\partial_z+\mathrm{i}\frac{qB_0}{2}r+\mathrm{i}g\mathrm{b}^1B_0\right)\right.\\&\left.-\mathrm{i}\sigma^3\partial_z\right]\psi_2\left(t,\vec{r}\right),\end{aligned} \tag{3.2a}$$

$$\begin{aligned}\mathrm{i}\partial_t\psi_2\left(t,\vec{r}\right)+2m\psi_2\left(t,\vec{r}\right)=&\left[-\mathrm{i}\sigma^1\left(\partial_r+\frac{1}{2r}\right)-\mathrm{i}\frac{\sigma^2}{\alpha r}\left(\partial_\varphi-\chi\partial_z+\mathrm{i}\frac{qB_0}{2}r+\mathrm{i}g\mathrm{b}^1B_0\right)\right.\\&\left.-\mathrm{i}\sigma^3\partial_z\right]\psi_1\left(t,\vec{r}\right).\end{aligned} \tag{3.2b}$$

Here the spinor $\psi_1\left(t,\vec{r}\right)$ is considered as the large component and the spinor $\psi_2\left(t,\vec{r}\right)$ is the small component. In the non-relativistic limit one usually assumes for the small component the relation $|2m\psi_2(t,\vec{r})|\gg|\mathrm{i}\partial_t\psi_2(t,\vec{r})|$. Hence, ignoring the time derivative on the left-hand side of the second equation above and substituting that approximate equation into the first one, finally results in the generalized Schrödinger-Pauli equation as[4]

$$\begin{aligned}\mathrm{i}\partial_t\psi_1=&-\frac{1}{2m}\left[\partial_r^2+\frac{1}{r}\partial_r+\frac{1}{\alpha^2r^2}\left(\partial_\varphi-\chi\partial_z\right)^2+\partial_z^2\right]\psi_1\\&+\frac{1}{8mr^2}\psi_1-\frac{\mathrm{i}}{m\alpha^2r^2}\left(\frac{qB_0}{2}r+g\mathrm{b}^1B_0\right)\left(\partial_\varphi-\chi\partial_z\right)\psi_1\\&+\frac{1}{2m\alpha^2r^2}\left(\frac{qB_0}{2}r+g\mathrm{b}^1B_0\right)^2\psi_1+\frac{\mathrm{i}\sigma^3}{2m\alpha r^2}\left(\partial_\varphi-\chi\partial_z\right)\psi_1\\&-\frac{\sigma^3}{2m\alpha r^2}\left(\frac{qB_0}{2}r+g\mathrm{b}^1B_0\right)\psi_1+\frac{qB_0\sigma^3}{4m\alpha r}\psi_1.\end{aligned} \tag{3.3}$$

The reader may realize that the operator on the right-hand-side of (3.3) is up to an additional factor $-1/2m$ identical to the one given in (2.13). Hence, we may simply repeat the same approach using the same ansatz (2.14) and arrive at the following radial wave equation

$$\begin{aligned}&\frac{\mathrm{d}^2\Phi_s(r)}{\mathrm{d}r^2}+\frac{1}{r}\frac{\mathrm{d}\Phi_s(r)}{\mathrm{d}r}+\frac{1}{r^2}\left[-\left(k^2-2m\mathcal{E}+\frac{q^2B_0^2}{4\alpha^2}\right)r^2\right.\\&\left.-\frac{qB_0\,\xi_\ell}{\alpha^2}r-\left(\frac{\xi_\ell^2}{\alpha^2}-\frac{s\,\xi_\ell}{\alpha}+\frac{1}{4}\right)\right]\Phi_s(r)=0.\end{aligned} \tag{3.4}$$

Obviously, Eq. (3.4) is identical in form with Eq. (2.15) when replacing $\mathcal{E}^2-m^2$ by $2m\mathcal{E}$. Thus, the solution of the wave equation (3.4) can be determined based on the generalized Laguerre polynomial

[4] Again we temporarily drop the t and $\vec{r}$ dependencies in the spinor component $\psi_1\left(t,\vec{r}\right)$.

as follows

$$\Phi^{\mathrm{NR}}_{sn\ell}(r) = \mathrm{N}^{\mathrm{NR}}_{n\ell}\, r^{-\frac{1}{2}\sqrt{1+\frac{4\xi_\ell^2}{\alpha^2}-\frac{4s\xi_\ell}{\alpha}}}\, \mathrm{e}^{-\sqrt{k^2-2m\mathcal{E}^{\mathrm{NR}}_{n\ell}+\frac{q^2B_0^2}{4\alpha^2}}\,r} \times \mathcal{L}_n^{\sqrt{1+\frac{4\xi_\ell^2}{\alpha^2}-\frac{4s\xi_\ell}{\alpha}}}\left(2\sqrt{k^2-2m\mathcal{E}^{\mathrm{NR}}_{n\ell}+\frac{q^2B_0^2}{4\alpha^2}}\,r\right), \tag{3.5}$$

where the superscript NR stands for the non-relativistic limit. Similarly, we can determine the non-relativistic energy eigenvalues corresponding to Eq. (3.4) as

$$\mathcal{E}^{\mathrm{NR}}_{sn\ell} = \frac{k^2}{2m} + \frac{q^2B_0^2}{8m\alpha^2} - \frac{B_0^2q^2\xi_\ell^2}{2m\alpha^4\left(1+2n+\sqrt{1+\frac{4\xi_\ell^2}{\alpha^2}-\frac{4s\xi_\ell}{\alpha}}\right)^2}. \tag{3.6}$$

Now, in order to investigate the non-relativistic limit of the energy eigenvalues in Eq. (2.19), we assume that $\mathcal{E}^{\mathrm{R}} \approx m + \mathcal{E}^{\mathrm{NR}}$ in the non-relativistic regime. In this way, if we take $\mathcal{E}^{\mathrm{R}} - m \approx \mathcal{E}^{\mathrm{NR}}$ and $\mathcal{E}^{\mathrm{R}} + m \approx 2m$, we arrive at $(\mathcal{E}^{\mathrm{R}})^2 - m^2 \approx 2m\,\mathcal{E}^{\mathrm{NR}}$. Accordingly, considering Eq. (2.19) and (3.6), we can clearly see that the energy eigenvalues given in Eq. (3.6) are the non-relativistic limit of the energy eigenvalues in Eq. (2.19). We will further comment on this observation in the next section where we show that in fact $(\mathcal{E}^{\mathrm{R}})^2 - m^2 = 2m\,\mathcal{E}^{\mathrm{NR}}$.

4 Conclusions and Comments

In this study we started with the description of the space-time containing curvature and torsion through the line element corresponding to the cosmic screw dislocation background and continued by providing a brief review of the mathematical relations corresponding to the spinor theory for such a geometric approach. Then, based on the interface between quantum mechanics and quantum field theory in the relativistic and non-relativistic regime, quantum dynamics of a charged Dirac fermion is studied under the background of the LSV effect described as a background having a privileged direction in space-time. Thus, the influence of the LSV scenario, which originates from a fixed space-like vector field in the direction of the coordinate r, generates a Kratzer-like potential in the background space-time described by the cosmic screw dislocation. In this regard, the interaction of a charged Dirac fermion, which has an induced electric dipole moment defined by the fixed space-like vector in the radial direction, with a uniform magnetic field in the direction of screw dislocation is investigated by using the relativistic and non-relativistic limit of the generalized Dirac equation in the presence of the Kratzer-like potential.

Furthermore, for both limits, the exact analytical solutions are obtained by solving the generalized Dirac equation and its non-relativistic limit, that is, the generalized Schrödinger-Pauli equation by the NU method in this background. In both cases the wave functions are represented in terms of the generalized Laguerre polynomials and the corresponding energy eigenvalues are obtained in terms of the parameters such as the electric charge q, the constant B_0 associated with the uniform magnetic field, the analogous electric dipole moment $\mathrm{d} = g\mathrm{b}^1$, the quantum numbers n and ℓ, the parameter α associated with the deficit angle, the rest mass m related to the fermionic field, the wave number k and the eigenvalues s of σ^3.

Obviously the discussion of the relativistic and non-relativistic case are very similar. In order to understand that in more detail let us put the Dirac eq. (2.11) in the form

$$\mathrm{i}\partial_t\Psi = \hat{H}_{\mathrm{D}}\Psi\,, \tag{4.1}$$

where the Dirac Hamiltonian can be represented by a 2×2 matrix of the form

$$\hat{H}_{\mathrm{D}} = \begin{pmatrix} m & Q \\ Q & -m \end{pmatrix} \tag{4.2}$$

with

$$Q := -\mathrm{i}\sigma^1\left(\partial_r + \frac{1}{2r}\right) - \mathrm{i}\frac{\sigma^2}{\alpha r}\left(\partial_\varphi - \chi\partial_z + \mathrm{i}\frac{qB_0}{2}r + \mathrm{i}g\mathrm{b}^1 B_0\right) - \mathrm{i}\sigma^3\partial_z = Q^\dagger \tag{4.3}$$

being a self-adjoint operator acting on $L^2(\mathbb{R}^3)\otimes\mathbb{C}^2$. For such Dirac Hamiltonians it is known, see Ref. [59] and references therein, that a unitary operator $\hat{U}$ exists, i.e. a Foldy Wouthuysen transformation, which brings it into a block diagonal form

$$\hat{H}_{\mathrm{FW}} = \hat{U}\hat{H}_{\mathrm{D}}\hat{U}^{-1} = \begin{pmatrix} \sqrt{Q^2+m^2} & 0 \\ 0 & -\sqrt{Q^2+m^2} \end{pmatrix} = \beta\otimes\sqrt{Q^2+m^2}$$

The Pauli Hamiltonian as defined in (3.3) reads in these terms

$$\hat{H}_{\mathrm{P}} = \frac{1}{2m}Q^2$$

Hence, we have

$$\hat{H}_{\mathrm{FW}} = \beta\otimes m\sqrt{1+2\hat{H}_{\mathrm{P}}/m}\,, \tag{4.4}$$

which shows that the Pauli Hamiltonian is indeed the non-relativistic limit of the Dirac Hamiltonian. Obviously the eigenvalues $\mathcal{E}^{\mathrm{R}}$ of the Dirac Hamiltonian and the eigenvalues $\mathcal{E}^{\mathrm{NR}}$ of the Pauli Hamiltonian are related by

$$\mathcal{E}^{\mathrm{R}} = \pm m\sqrt{1+\frac{2\mathcal{E}^{\mathrm{NR}}}{m}}$$

which for large m and taking the upper sign results in $\mathcal{E}^{\mathrm{R}} \simeq m + \mathcal{E}^{\mathrm{NR}}$. It is also obvious that the eigenstates of the $\hat{H}_{\mathrm{D}}$ are closely related to those of $\hat{H}_{\mathrm{P}}$ via $\hat{U}$. Finally, let us note that the Dirac Hamiltonian (4.2) exhibits a quantum mechanical SUSY structure according to the general approach of Ref. [59] with Q in essence representing the associated SUSY charge.

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Data Availability

The manuscript has no associated data.

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